Computational complexity theory for spaces of integrable functions

Florian Steinberg

Technische Universität Darmstadt

April 15, 2016

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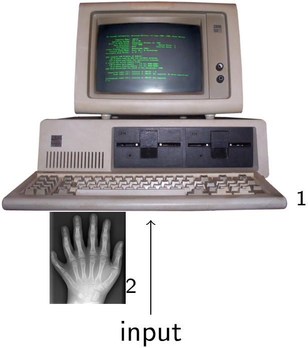
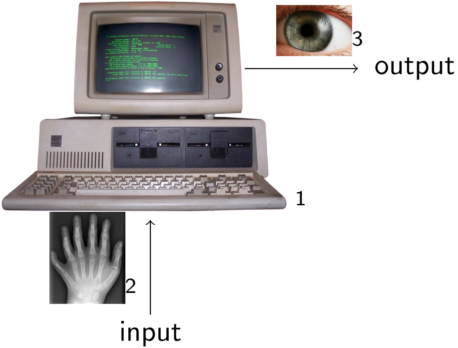
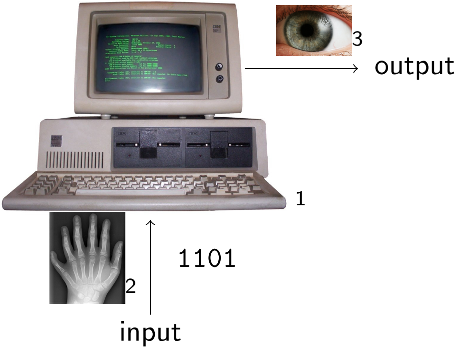
Programs with subroutine calls

 [1]


1
2
input


3
output
1
2
input



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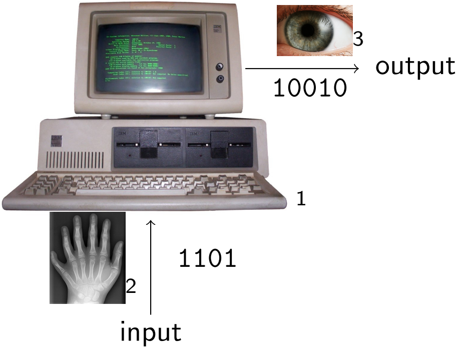
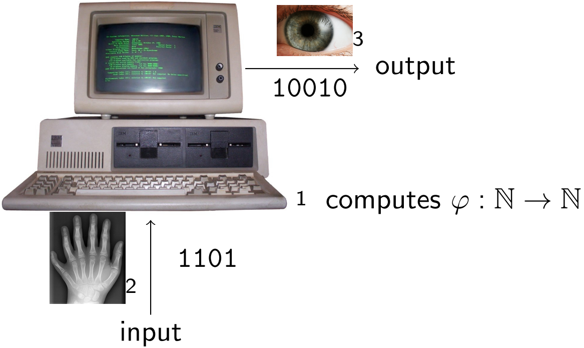

3
output
10010
1
1101
2
input



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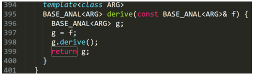
Programs with subroutine calls


```
    template<class ARG>
    BASE_ANAL<ARG> derive(const BASE_ANAL<ARG>& f) {
      BASE_ANAL<ARG> g;
      g = f;
      g.derive();
      return g;
    }
  }
```


[3]

10010 → output


[1] computes $\varphi : \mathbb{N} \to \mathbb{N}$


[2]

1101 ↑ input

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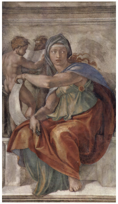

oracle


[1] computes $\varphi : \mathbb{N} \to \mathbb{N}$


[3]

$\xrightarrow{10010}$ output


[2]

1101 $\uparrow$ input

---

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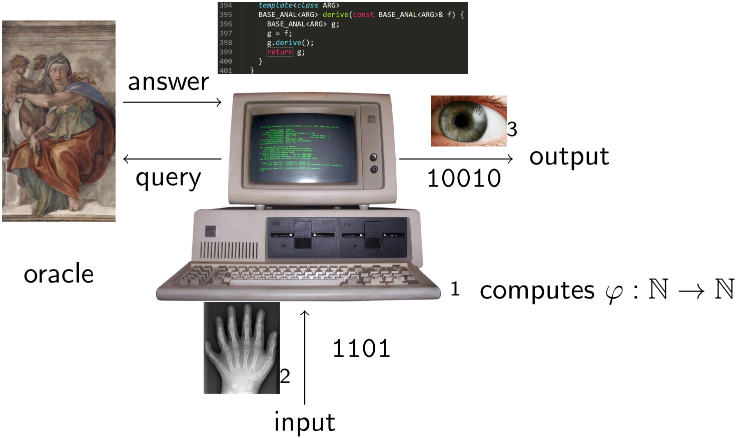

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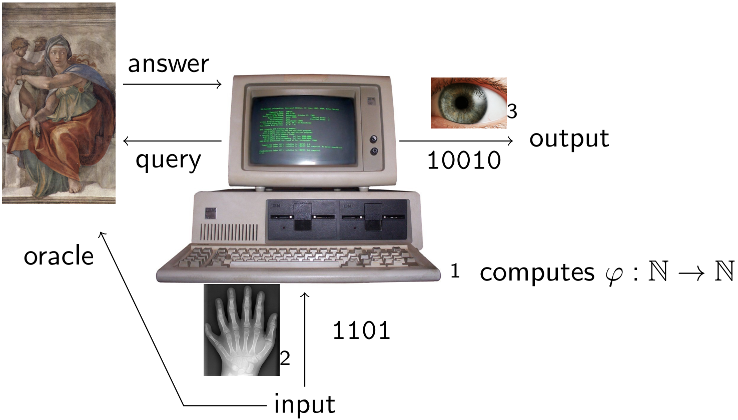

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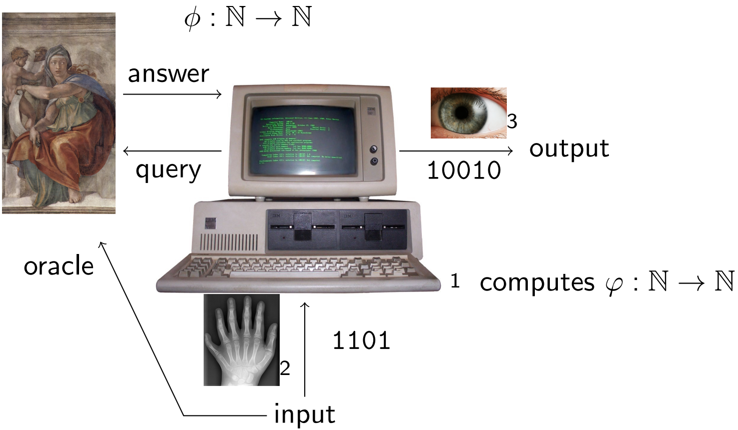

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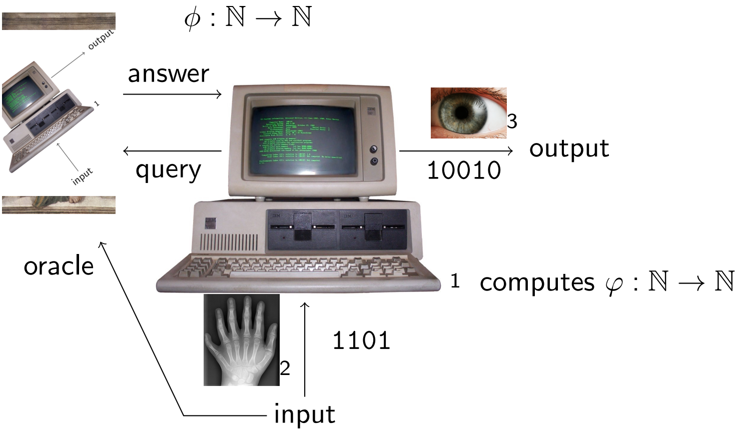

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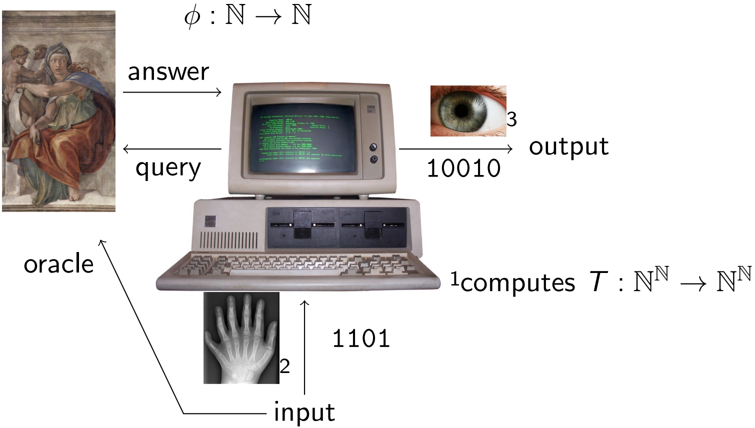

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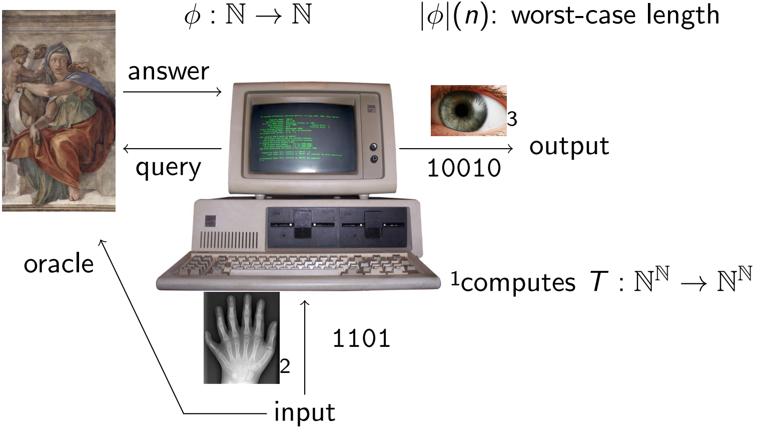
$\phi : \mathbb{N} \to \mathbb{N}$ $\quad |\phi|(n)$: worst-case length



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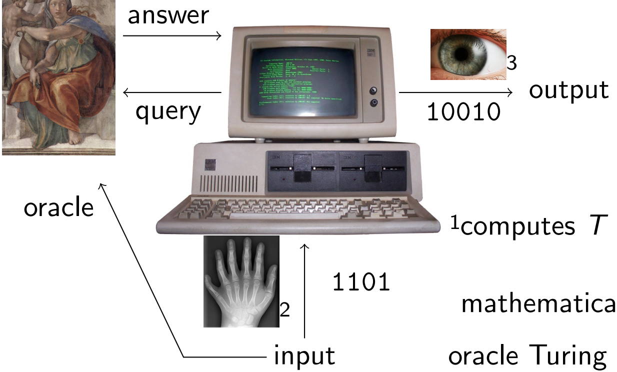

[1]computes $T : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

mathematical model:

oracle Turing machines

---

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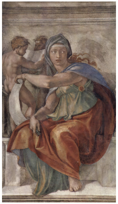
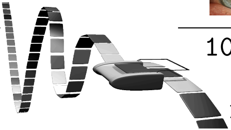


oracle

answer $\longrightarrow$

$\longleftarrow$ query

[1]

[3]

$\longrightarrow$ output

10010

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1101 $\uparrow$

input

computes $T : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

mathematical model:

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Realistic model:

- Equality undecidable.

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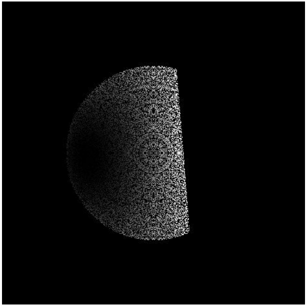

Realistic model:

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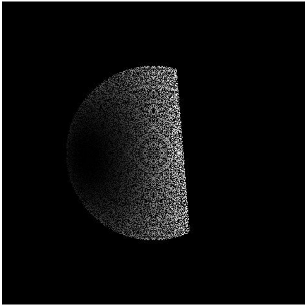

Realistic model:

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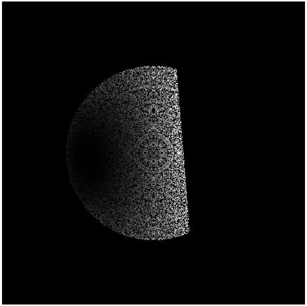

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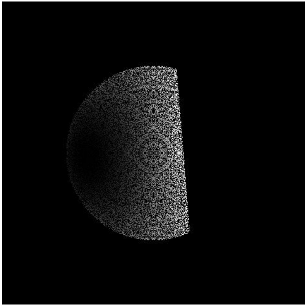

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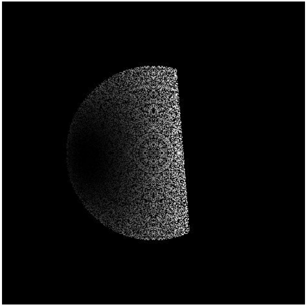

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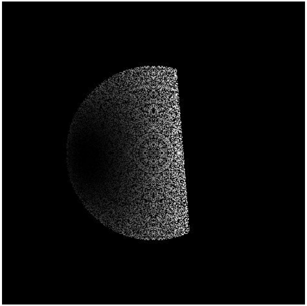
Computability theory well developed in metric spaces.



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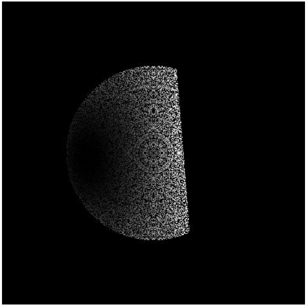

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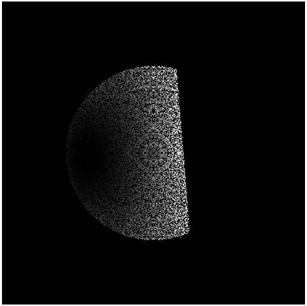

Friedman and Ko (82/84):

- Maximizing a function $\leftrightarrow$ $\mathcal{P}$ vs. $\mathcal{NP}$.

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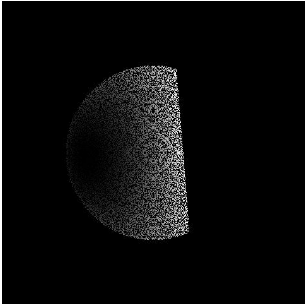

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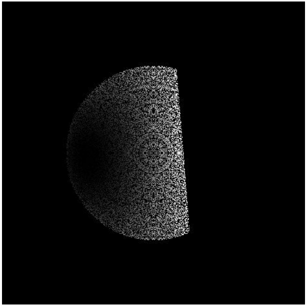

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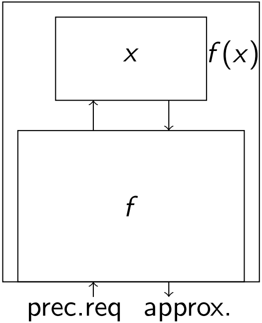

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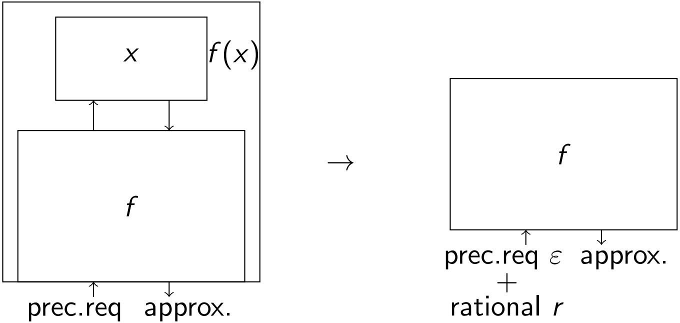

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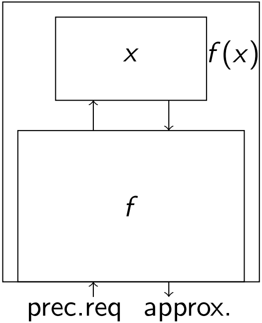
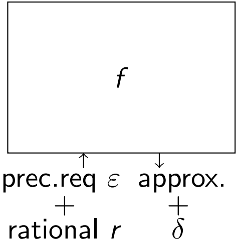


$\rightarrow$

$$|x-y| \leq \delta \Rightarrow |f(x)-f(y)| < \varepsilon$$



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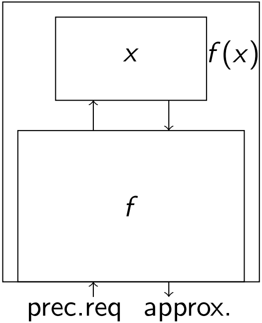

$\leftrightarrow$

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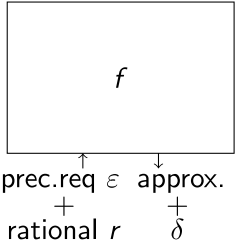
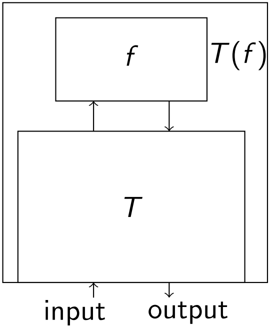
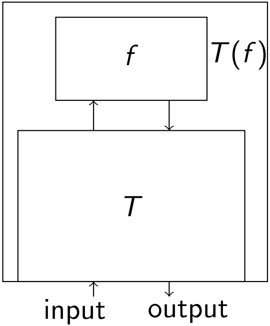



f
T(f)
T
input
output

Theorem (Kawamura and Cook (2013))

This is the least set about information about a function such that evaluation is fast.



Theorem (Kawamura and Cook (2013))

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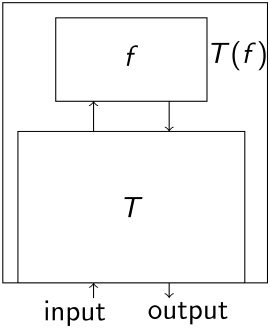
Theorem (folklore (2003); formally: Ziegler et al (2014))

The norm on $C([0,1])$ is exponential- but not polynomial-time computable.



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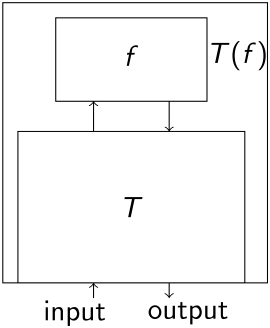
Theorem ("")

Integration is exponential- but not polynomial-time computable.



Theorem (Pour-El, Richards 1989)

There is a computable function f such that the solution of the wave equation

$$\Delta u = \frac{\partial^2 u}{\partial t^2}$$

$$u(0) = f$$

$$\frac{\partial u}{\partial t} = 0$$

is not computable at time 1.

$L^p([0,1])$:

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Functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_p := \int_0^1 |f(t)|^p dt < \infty$$

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(if we identify functions that coincide almost everywhere).

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$L^\infty([0,1])$: bounded functions with supremum norm.

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$C([0,1])$

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$C([0,1]) \subsetneq L^\infty([0,1])$

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$$C([0,1]) \subsetneq L^\infty([0,1]) \subsetneq L^p([0,1])$$

$L^p([0,1])$:

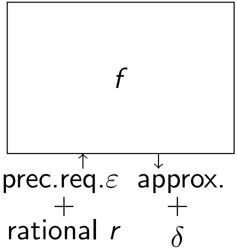
Functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} < \infty$$

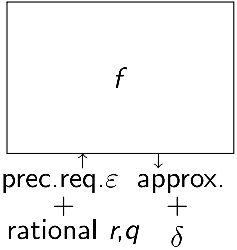
$\|f\|_p$ is a norm and $L^p([0,1])$ a Banach space.
(if we identify functions that coincide almost everywhere).

$L^\infty([0,1])$: bounded functions with supremum norm.

$$C([0,1]) \subsetneq L^\infty([0,1]) \subsetneq L^p([0,1]) \subsetneq L^1([0,1])$$



Consider $f \in L^1([0,1])$



approx. to $\int_r^q f(t)dt$.

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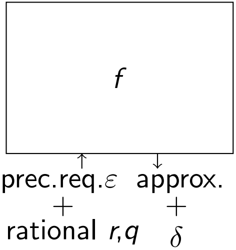
$$|h| \leq \delta \Rightarrow \|\tilde{f} - \tau_h \tilde{f}\|_1 < \varepsilon$$



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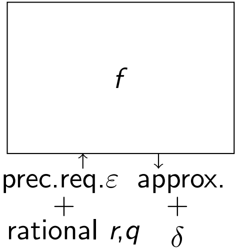

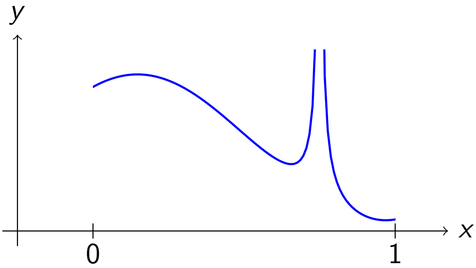

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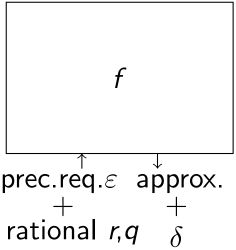

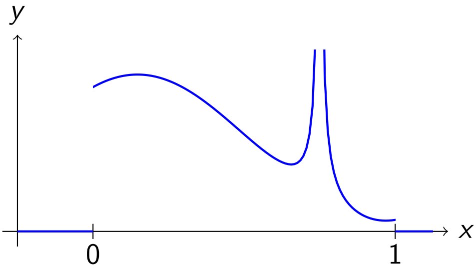

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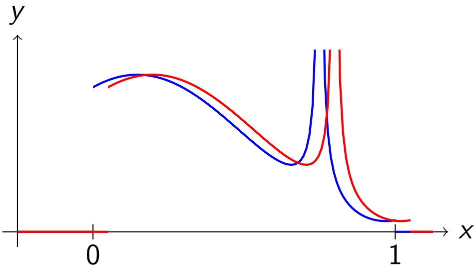

f

prec.req.ε + rational r,q → approx. + δ

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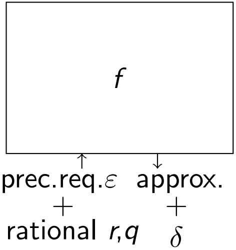

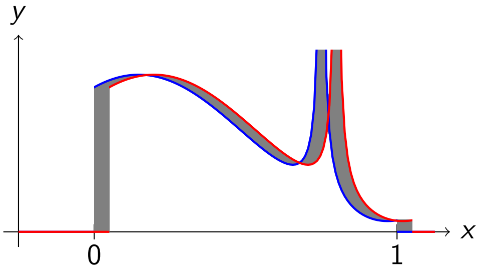

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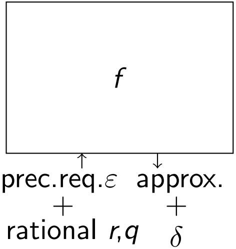

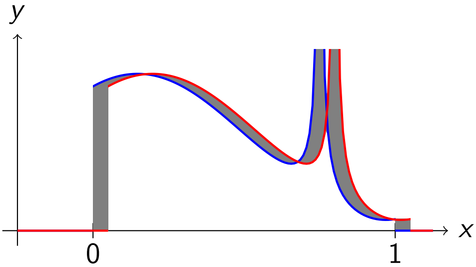

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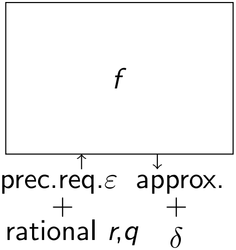

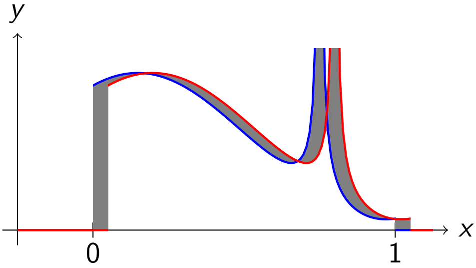

L^p-modulus of f $\rightsquigarrow$ Frechet Kolmogorov for bounded sets.

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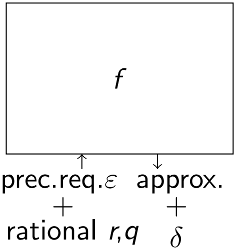
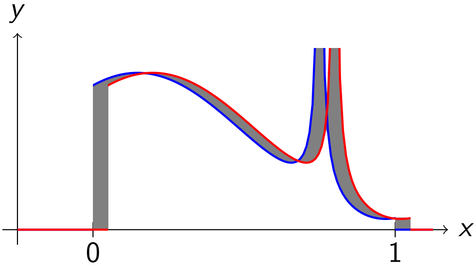




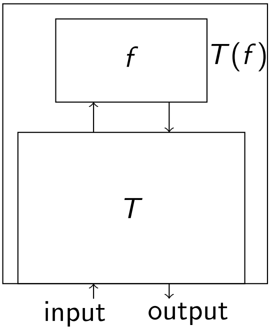
L^p-modulus of $f \rightsquigarrow$ Frechet Kolmogorov for bounded sets.

approx. to $\int_r^q f(t)dt$.
Also: higher dimensions

Consider $f \in L^p([0,1])$

Theorem

Integration is possible in polynomial time.



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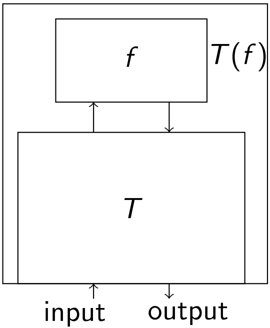
The norm on L^p is exponential but not polynomial time computable.



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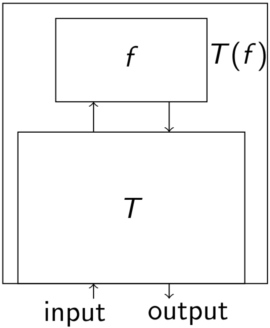
Theorem

Not the smallest set of information to allow fast integration.



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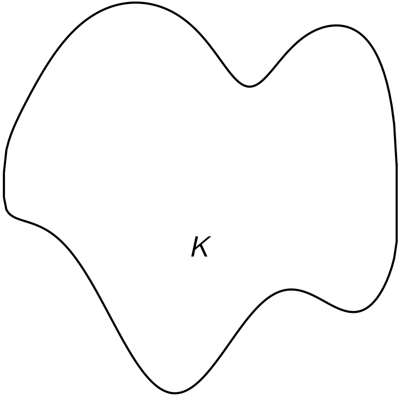

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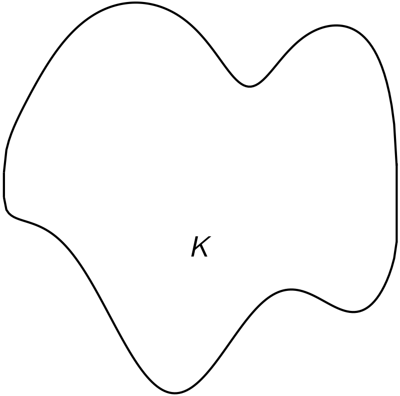
$\mathcal{N}_\varepsilon(K)$: # of ε-Balls to cover.



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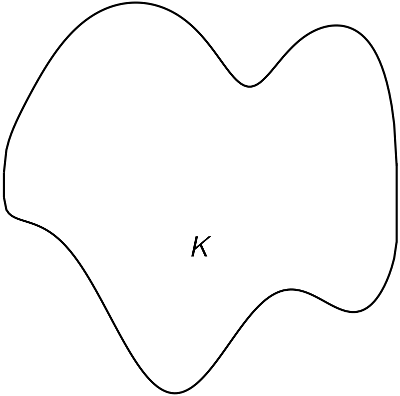

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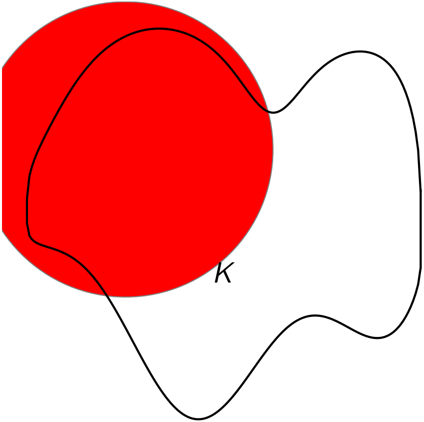

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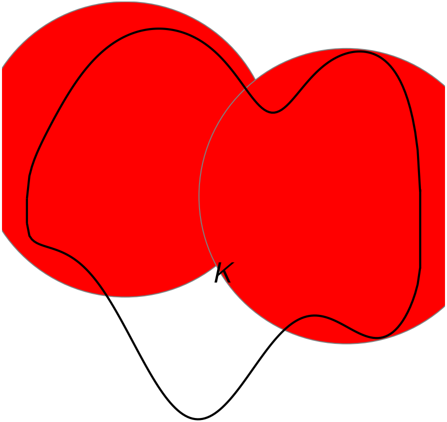

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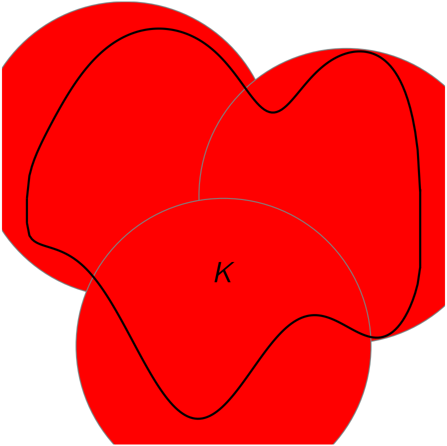

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$\mathcal{N}_\varepsilon(K)$: # of ε-Balls to cover.

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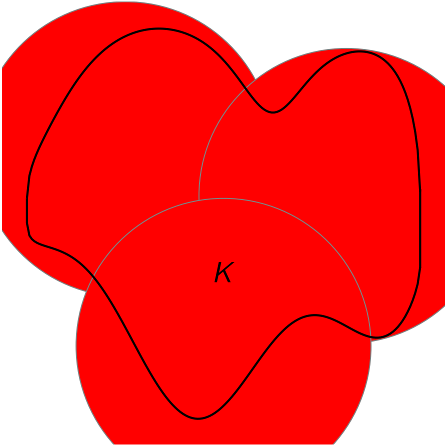

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$\mathcal{N}_\varepsilon(K)$: # of ε-Balls to cover.

$$\mathcal{N}_1(K) \leq 3$$

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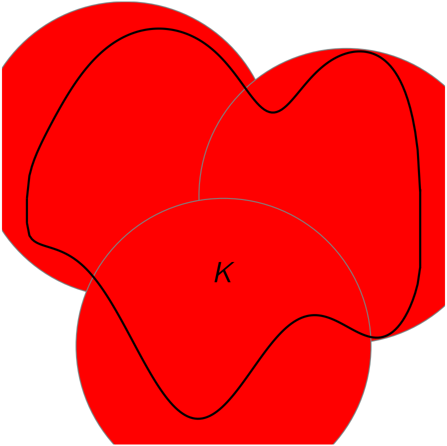

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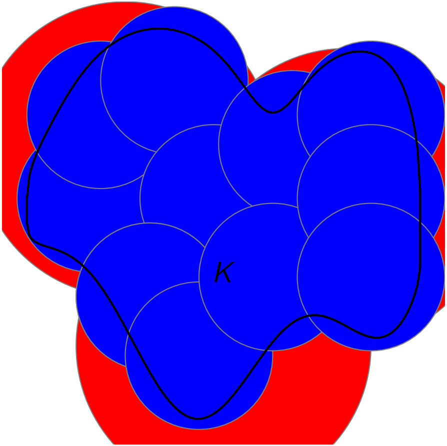

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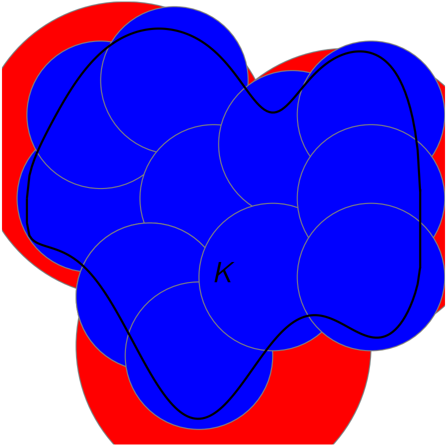
$|K| : \mathbb{N} \to \mathbb{N}$
length of $\varepsilon \mapsto \mathcal{N}_\varepsilon(K)$.



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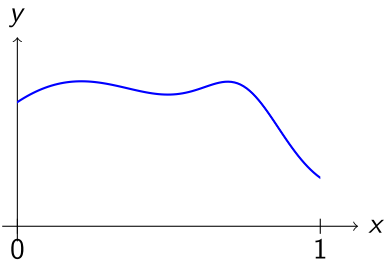

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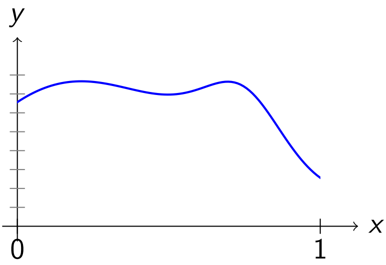

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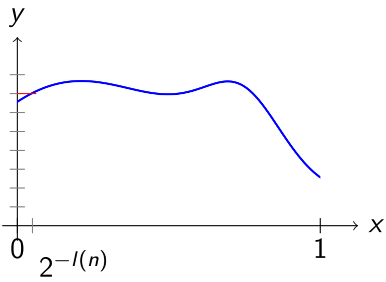

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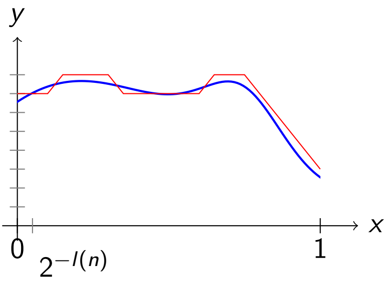

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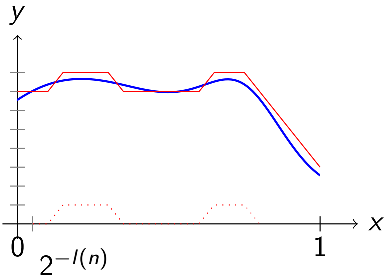

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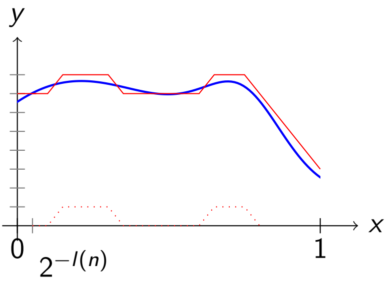

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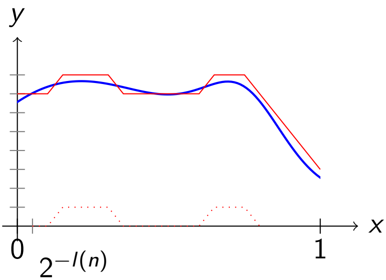

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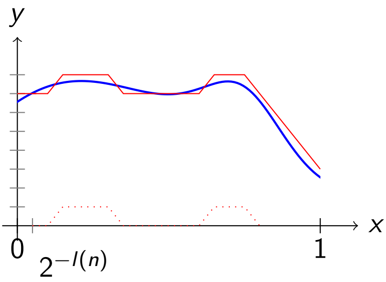

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$$2^{l(n-3)} \lesssim |K^p_l|(n) \lesssim 2^{l(n+2)+l(0)} + n + l(0)$$

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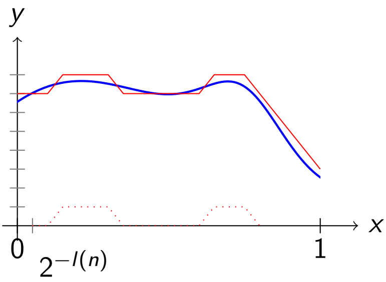

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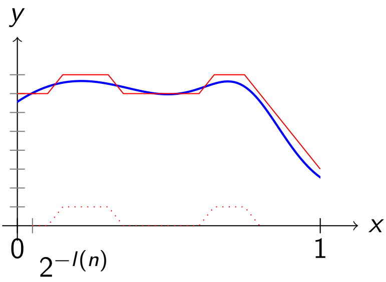

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- $\rightsquigarrow$ use previous Theorem.
- $\rightsquigarrow$ use coding theory.

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Corollary

The set of functions getting a short name in the encoding of L^p is so big that no other encoding of it computes the metric faster.

$W^{1,p}$

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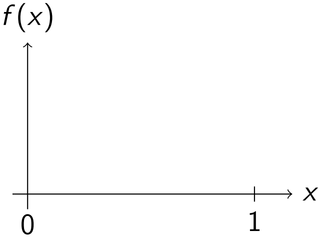

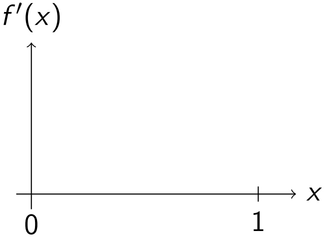

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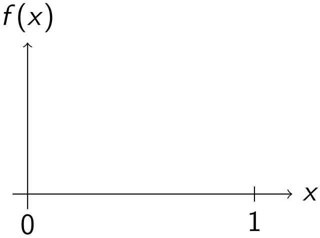
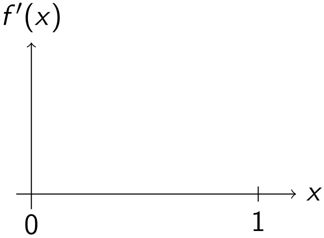
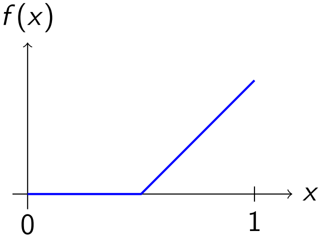




Sobolev spaces

$$W^{1,p} := \{f \in L^p \mid f' \in L^p\}, \quad \int f' u d\lambda = -\int f u' \lambda$$



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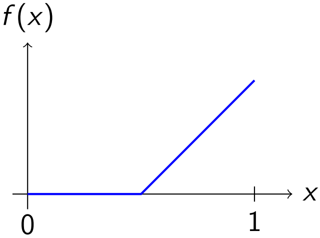

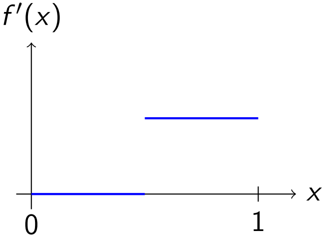

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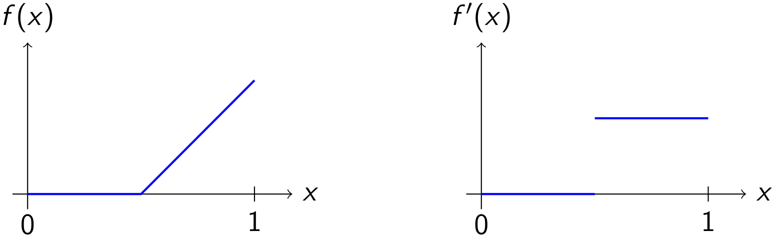

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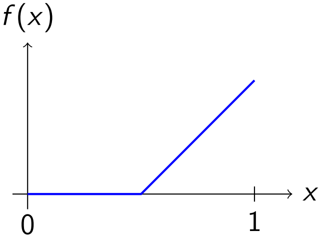

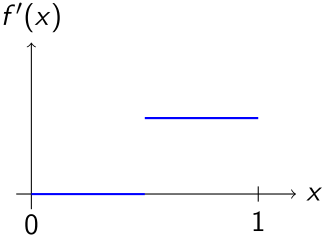

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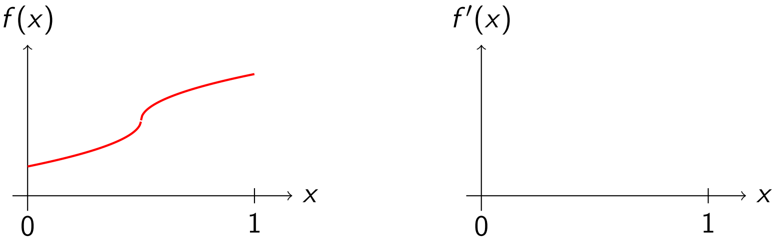

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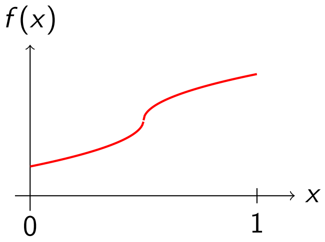

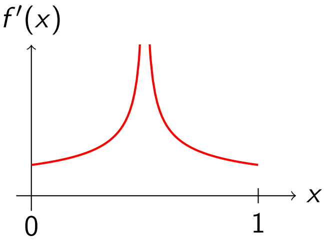

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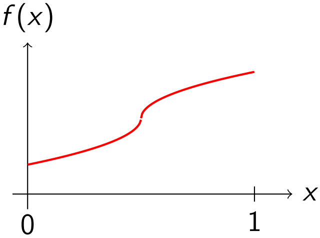

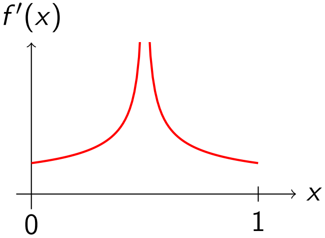

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$W^{m,p}$: replace L^p-mod. of f by L^p-mod of $f^{(m)}$.

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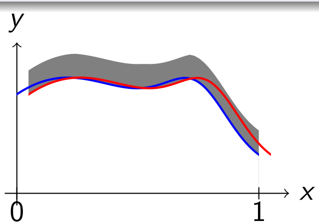

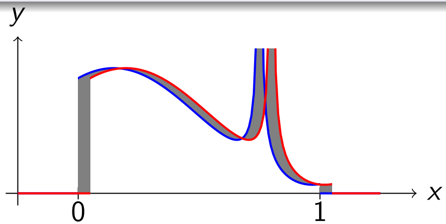

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- *The inclusion* $W^{m,p} \hookrightarrow W^{m-1,p}$.
- *Differentiation* $\frac{d}{dx} : W^{m,p} \to W^{m-1,p}, \quad f \mapsto f'$.

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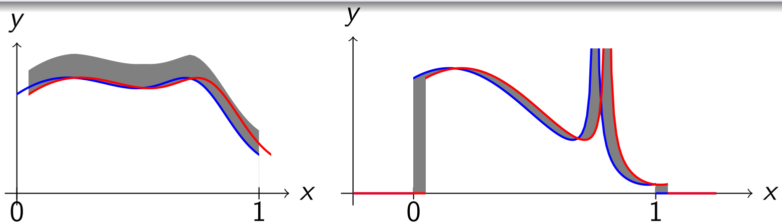

Thanks!